

Gauge transformation of quantum states in probability representation

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Abstract

The gauge invariance of the evolution equations of tomographic probability distribution functions of quantum particles in an electromagnetic field is illustrated. Explicit expressions for the transformations of ordinary tomograms of states under a gauge transformation of electromagnetic field potentials are obtained. Gauge-independent optical and symplectic tomographic quasi-distributions and tomographic probability distributions of states of quantum system are introduced, and their evolution equations having the Liouville equation in corresponding representations as the classical limit are found.

Keywords: Quantum tomography, gauge invariance, evolution equation, optical tomogram, symplectic tomogram.

1 Introduction

Gauge invariance is a fundamental quality of classical field theory and quantum electrodynamics [1, 2], as well as of Yang-Mills theory [3]. In quantum mechanics the gauge transformation makes the specific change [4] of the wave function [5] phase.

For the formulation of quantum mechanics in phase space many scientists suggested different kinds of quasidistributions to represent the quantum states. For example, Wigner function [6], Blohintsev function [7], Glauber-Sudarshan P -function [8, 9], Husimi Q -function [10] can be effectively used to formulate the quantum evolution and obtain the energy levels of quantum states. All these quasidistributions are related to wave function or density matrix by integral transformations.

In Ref. [11] the probability representation of quantum mechanics was introduced (see, e.g. [12]), in which the quantum states are described by fair probabilities, called quantum tomograms. Different kinds of tomograms, e.g., optical tomograms [13, 14], symplectic tomograms [15], spin-tomograms [16, 17] give the realization of star-product quantization schemes based on existence of specific quantizer and dequantizer operators [18, 19]. The star-product schemes bring about the constructions for non-commutative algebra of Wigner-Weil symbols of operators acting on a Hilbert space (see, e.g., [20, 21, 22]).

The evolution equation and energy spectrum equation for optical tomogram were obtained in [23]. The evolution equation for symplectic tomogram was obtained in [11]. On the other hand, the gauge properties known for Schrödinger equation for wave function and Moyal equation for the Wigner function [24] have not been considered till now in the tomographic representation of quantum mechanics.

The aim of our paper is to explore the gauge properties of quantum tomograms, including the star-product aspects, to introduce the gauge-independent tomograms, and to obtain evolution equations of quantum states in gauge-independent tomographic representations.

To begin with let us recall how the gauge invariance of non-relativistic quantum mechanics is realized in the wave function or density matrix representation, and remind basic formulas of conversion of wave function and density matrix of a particle under the gauge transformation of the potentials of the electromagnetic field.

Consider the motion of a quantum particle having a spin in the electromagnetic field with the vector potential $\mathbf{A}(\mathbf{q}, t)$ and the scalar potential $\varphi(\mathbf{q}, t)$. As it is known, the Hamiltonian of such a system has the form [4]

$$\hat{H} = \frac{1}{2m} \left(\hat{\mathbf{P}} - \frac{e}{c} \mathbf{A} \right)^2 + e\varphi - \hat{\boldsymbol{\kappa}} \mathbf{B}, \quad (1)$$

where $\hat{\mathbf{P}} = -i\hbar\partial/\partial\mathbf{q}$ is a generalized momentum operator, m and e are mass and charge of the particle, $\mathbf{B} = \text{rot}\mathbf{A}$ is a magnetic field strength, $\hat{\boldsymbol{\kappa}}$ is an operator of quantum-mechanical magnetic moment

$$\hat{\boldsymbol{\kappa}} = \frac{\kappa}{s}\hat{\mathbf{s}}, \quad (2)$$

where s is a spin of the particle, $\hat{\mathbf{s}}$ is a spin operator, and κ is a constant characteristic of the particle (the value of the intrinsic magnetic moment) that is the highest possible modulo value κ_z of projection of the magnetic moment on the z axis achieved with the projection of the spin on this axis equal to s .

From the classical electrodynamics it is known that potentials of the field are defined only up to the gauge transformation [1]

$$\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi, \quad \varphi \rightarrow \varphi - \frac{1}{c}\frac{\partial\chi}{\partial t}, \quad (3)$$

where χ is an arbitrary function of spatial coordinates and time.

Since the electric field intensity \mathbf{E} and the magnetic field strength \mathbf{B} are defined in terms of the potentials as:

$$\mathbf{E} = -\text{grad}\varphi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}, \quad \mathbf{B} = \text{rot}\mathbf{A}, \quad (4)$$

then the gauge transformation (3) does not affect the values of \mathbf{E} and \mathbf{B} . Therefore the part of Hamiltonian (1) responsible for the interaction of the spin with the magnetic field is independent on the gauge transformation.

The requirement of invariance of the Schrödinger equation under the gauge transformation simultaneously with the gauge-independence of “probability density” $|\Psi|^2$ leads us to the form of the conversion of the wave function [4]:

$$\Psi \rightarrow \Psi \exp\left(\frac{ie}{c\hbar}\chi\right). \quad (5)$$

Accordingly, the conversions of the density matrix of the state and the Hamiltonian of the system under the gauge transformation acquire the forms:

$$\hat{\rho}_c = \exp\left(\frac{ie}{c\hbar}\chi\right)\hat{\rho}\exp\left(-\frac{ie}{c\hbar}\chi\right), \quad (6)$$

$$\hat{H}_c = \exp\left(\frac{ie}{c\hbar}\chi\right)\hat{H}\exp\left(-\frac{ie}{c\hbar}\chi\right), \quad (7)$$

and the von-Neumann equation is also invariant under transformations (3), (6)

$$i\hbar\frac{\partial}{\partial t}\hat{\rho} = [\hat{H}, \hat{\rho}] \rightarrow i\hbar\frac{\partial}{\partial t}\hat{\rho}_c = [\hat{H}_c, \hat{\rho}_c]. \quad (8)$$

The paper is organized as follows. In Sec. 2 we find transformations of ordinary quantum tomograms in general case in terms of gauge-independent quantizer and dequantizer operators. In Sec. 3 we obtain the evolution equations for classical and quantum particles in the classical electro-magnetic field in tomographic representations with gauge-independent dequantizers, we discuss the gauge invariance of these equations and illustrate that the quantum tomographic equations do not be transformed to the corresponding classical equations when $\hbar \rightarrow 0$. In Sec. 4 we introduce gauge-independent optical and symplectic quantum tomographic quasidistributions and derive evolution equations for such representations. In Sec. 5 we introduce and study the gauge-independent tomographic probability representation and find the evolution equation for it. Conclusion is presented in Sec. 6.

2 Gauge transformations of ordinary quantum tomograms

In the probability representation of quantum mechanics the states of the system are described by a probability distribution functions $w(z, \eta, t)$ called quantum tomograms, where z is a set of distribution variables, η is a set of parameters of corresponding tomography, and t is time. According to the universal star-product scheme (see [21]), the tomograms are introduced as the average values of dequantizer operators $\hat{U}(z, \eta)$,

$$w(z, \eta, t) = \text{Tr} \left\{ \hat{\rho}(t) \hat{U}(z, \eta) \right\}, \quad (9)$$

The inverse transformation is determined by the quantizer operator $\hat{D}(z, \eta)$

$$\hat{\rho}(t) = \int \hat{D}(z, \eta) w(z, \eta, t) dz d\eta. \quad (10)$$

The von-Neumann equation in the tomographic representation has the form [25]:

$$\frac{\partial}{\partial t} w(z, \eta, t) = \frac{2}{\hbar} \int \text{Im} \left[\text{Tr} \left\{ \hat{H}(t) \hat{D}(z', \eta') \hat{U}(z, \eta) \right\} \right] w(z', \eta', t) dz' d\eta'. \quad (11)$$

It is easy to see that if we determine in definition (9) that the dequantizer and the quantizer are gauge-independent, then equation (11) is invariant under the gauge transformation only with the following transformation of tomograms:

$$\begin{aligned} w(z, \eta, t) &\rightarrow w_c(z, \eta, t) = \text{Tr} \left\{ \exp \left(\frac{ie}{c\hbar} \chi \right) \hat{\rho}(t) \exp \left(-\frac{ie}{c\hbar} \chi \right) \hat{U}(z, \eta) \right\} \\ &= \text{Tr} \left\{ \exp \left(\frac{ie}{c\hbar} \chi \right) \int \hat{D}(z', \eta') w(z', \eta', t) dz' d\eta' \exp \left(-\frac{ie}{c\hbar} \chi \right) \hat{U}(z, \eta) \right\}. \end{aligned} \quad (12)$$

Introducing the notation for the kernel $G(z, \eta, z', \eta')$

$$G(z, \eta, z', \eta') = \text{Tr} \left\{ \exp \left(\frac{ie}{c\hbar} \chi \right) \hat{D}(z', \eta') \exp \left(-\frac{ie}{c\hbar} \chi \right) \hat{U}(z, \eta) \right\}, \quad (13)$$

for the gauge transformed function $w_c(z, \eta, t)$ we get

$$w_c(z, \eta, t) = \int G(z, \eta, z', \eta') w(z', \eta', t) dz' d\eta'. \quad (14)$$

Thus, under the gauge transformation of the electromagnetic field potentials the tomogram of the state is converted by means of integral transformation (14), in which the explicit form of the kernel depends on the type of tomography.

If we have spinless quantum particle with mass m in three-dimensional space, then the dequantizer and the quantizer for the optical tomography have the form [26]

$$\hat{U}_w(\mathbf{X}, \boldsymbol{\theta}) = \prod_{\sigma=1}^3 \delta \left(X_{\sigma} - \hat{q}_{\sigma} \cos \theta_{\sigma} - \hat{P}_{\sigma} \frac{\sin \theta_{\sigma}}{m\omega_{\sigma}} \right), \quad (15)$$

$$\hat{D}_w(\mathbf{X}, \boldsymbol{\theta}) = \int \prod_{\sigma=1}^3 \frac{\hbar |\eta_{\sigma}|}{2\pi m\omega_{\sigma}} \exp \left\{ i\eta_{\sigma} \left(X_{\sigma} - \hat{q}_{\sigma} \cos \theta_{\sigma} - \hat{P}_{\sigma} \frac{\sin \theta_{\sigma}}{m\omega_{\sigma}} \right) \right\} d^3\eta, \quad (16)$$

where \hat{P}_{σ} are components of the generalized momentum operator and ω_{σ} are constants that have the dimension of frequency. Further for simplicity we choose the set $\{\omega_{\sigma}\}$ so that $\omega_1 = \omega_2 = \omega_3 = \omega$.

Substituting these expressions of the dequantizer and the quantizer to equation (13), after some calculations using the formula for the matrix elements

$$\langle q'_\sigma | e^{i(a\hat{q}_\sigma + b\hat{P}_\sigma)} | q_\sigma \rangle = e^{ia(q_\sigma + q'_\sigma)/2} \delta(q'_\sigma - q_\sigma + b), \quad (17)$$

we obtain

$$\begin{aligned} G_w(\mathbf{X}, \boldsymbol{\theta}, \mathbf{X}', \boldsymbol{\theta}') &= \frac{1}{(4\pi^2\hbar)^3} \int \exp \left\{ \frac{ie}{c\hbar} \left[\chi \left(\frac{k_\sigma \sin \theta_\sigma}{m\omega} + \frac{\sqrt{\hbar} \sin \theta'_\sigma}{2\sqrt{m\omega}} r_\sigma \right) - \chi \left(\frac{k_\sigma \sin \theta_\sigma}{m\omega} - \frac{\sqrt{\hbar} \sin \theta'_\sigma}{2\sqrt{m\omega}} r_\sigma \right) \right] \right\} \\ &\times \prod_{\sigma=1}^3 |r_\sigma| \exp \left\{ ir_\sigma \sqrt{\frac{m\omega}{\hbar}} \left(X'_\sigma - X_\sigma \frac{\sin \theta'_\sigma}{\sin \theta_\sigma} \right) - ik_\sigma r_\sigma \frac{\sin(\theta - \theta')}{\sqrt{m\omega\hbar}} \right\} d^3k d^3r. \end{aligned} \quad (18)$$

Further simplification of this expression can, unfortunately, be possible only with the explicit expression for the function χ .

For the spinless symplectic tomography dequantizer and quantizer are given by formulas

$$\hat{U}_M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \prod_{\sigma=1}^3 \delta(X_\sigma - \hat{q}_\sigma \mu_\sigma - \hat{P}_\sigma \nu_\sigma), \quad (19)$$

$$\hat{D}_M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \prod_{\sigma=1}^3 \frac{m\omega}{2\pi} \exp \left\{ i\sqrt{\frac{m\omega}{\hbar}} (X_\sigma - \hat{q}_\sigma \mu_\sigma - \hat{P}_\sigma \nu_\sigma) \right\}, \quad (20)$$

and from (13) we can obtain

$$\begin{aligned} G_M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}, \mathbf{X}', \boldsymbol{\mu}', \boldsymbol{\nu}') &= \left(\frac{m\omega}{4\pi^2\hbar} \right)^3 \int \exp \left\{ \frac{ie}{c\hbar} \left[\chi \left(\nu_\sigma k_\sigma + \frac{\sqrt{m\omega\hbar}}{2} \nu'_\sigma \right) - \chi \left(\nu_\sigma k_\sigma - \frac{\sqrt{m\omega\hbar}}{2} \nu'_\sigma \right) \right] \right\} \\ &\times \prod_{\sigma=1}^3 \exp \left\{ i\sqrt{\frac{m\omega}{\hbar}} \left[k_\sigma (\mu \nu'_\sigma - \mu'_\sigma \nu) + X'_\sigma - X_\sigma \frac{\nu'_\sigma}{\nu_\sigma} \right] \right\} d^3k. \end{aligned} \quad (21)$$

Note, that the kernels G_w and G_M are connected by the relation

$$G_w(\mathbf{X}, \boldsymbol{\theta}, \mathbf{X}', \boldsymbol{\theta}') = \int \frac{|r_1| |r_2| |r_3|}{(m\omega)^3} G_M \left(X_\sigma, \cos \theta_\sigma, \frac{\sin \theta_\sigma}{m\omega}, r_\sigma X', r_\sigma \cos \theta'_\sigma, r_\sigma \frac{\sin \theta'_\sigma}{m\omega} \right) d^3r.$$

Consider now the positive vector non-redundant tomography of the particle with spin [27], [28]

$$\mathbf{w}(z, \eta, t) = \text{Tr} \left\{ \hat{\rho}(t) \hat{\mathbf{U}}(z, \eta) \right\}, \quad (22)$$

where the trace is calculated also over spin indexes. In this representation the components of the dequantizer and the quantizer are defined by formulas

$$\hat{U}_{j(nl)}(z, \eta) = \hat{U}(z, \eta) \otimes \hat{\mathcal{U}}_{j(nl)}, \quad \hat{D}_{(nl)j}(z, \eta) = \hat{D}(z, \eta) \otimes \hat{\mathcal{D}}_{(nl)j}, \quad (23)$$

where $\hat{\mathcal{U}}_{j(nl)}$ and $\hat{\mathcal{D}}_{(nl)j}$ are spin dequantizer and quantizer, $j = \overline{1, (2s+1)^2}$ is the index corresponding to the j th component of the vector tomogram $\mathbf{w}(z, \eta, t)$, and $n, l = \overline{1, (2s+1)}$ are spin indexes. Since

$$\sum_{n,l=1}^{2s+1} \hat{\mathcal{D}}_{(nl)j'} \hat{\mathcal{U}}_{j(nl)} = \delta_{jj'}, \quad (24)$$

then, according to general formula (13), the kernel of the transformation of the vector tomogram $\mathbf{w}(z, \eta, t)$ takes the form:

$$G_{jj'}(z, \eta, z', \eta') = \text{Tr} \left\{ \exp \left(\frac{ie}{\hbar c} \chi \right) \hat{D}(z', \eta') \otimes \hat{D}_{j'} \exp \left(-\frac{ie}{\hbar c} \chi \right) \hat{U}(z, \eta) \otimes \hat{U}_j \right\} = \delta_{jj'} G(z, \eta, z', \eta'), \quad (25)$$

i.e., the vector tomogram $\mathbf{w}(z, \eta, t)$ under the gauge transformation is converted by components through the integral transformation:

$$\mathbf{w}(z, \eta, t) \rightarrow \mathbf{w}_c(z, \eta, t) = \int G(z, \eta, z', \eta') \mathbf{w}(z', \eta', t) dz' d\eta', \quad (26)$$

where $G(z, \eta, z', \eta')$ is a kernel of the integral transformation for the spinless case. This formula is valid for arbitrary spin.

Thus we see that if the dequantizer is gauge-independent, then the tomogram is gauge-dependent, and the evolution equation is gauge-invariant but gauge-dependent.

3 Gauge invariance of evolution equations

Let us consider in more detail the gauge invariance of the quantum evolution equations in the tomographic representations with gauge-independent dequantizers and the question of limiting transition of such equations to classics when $\hbar \rightarrow 0$. At first, we will get the Liouville equation in the electro-magnetic field in the tomographic representations.

For the classical ensemble of non-interacting particles with mass m and charge e this equation in the phase space has the form:

$$\frac{\partial}{\partial t} W_{\text{cl}}(\mathbf{q}, \mathbf{p}, t) + \frac{\mathbf{p}}{m} \frac{\partial}{\partial \mathbf{q}} W_{\text{cl}}(\mathbf{q}, \mathbf{p}, t) + e \left(\mathbf{E}(\mathbf{q}, t) + \frac{1}{mc} [\mathbf{p} \times \mathbf{B}(\mathbf{q}, t)] \right) \frac{\partial}{\partial \mathbf{p}} W_{\text{cl}}(\mathbf{q}, \mathbf{p}, t) = 0, \quad (27)$$

where \mathbf{p} is a kinetic momentum, $\mathbf{E}(\mathbf{q}, t)$ and $\mathbf{B}(\mathbf{q}, t)$ are electric and magnetic fields, defined by the formulas (4), $W_{\text{cl}}(\mathbf{q}, \mathbf{p}, t)$ is a distribution function of non-interacting particles.

The distribution function $W_{\text{cl}}(\mathbf{q}, \mathbf{p}, t)$ is independent on the gauge transformation [1], because the Liouville equation (27) includes only gauge-independent intensities of the electro-magnetic field. Consequently, the optical and symplectic tomograms of the function $f(\mathbf{q}, \mathbf{p}, t)$ defined by the formulas [23]

$$w_{\text{cl}}(\mathbf{x}, \boldsymbol{\theta}, t) = \int W_{\text{cl}}(\mathbf{q}, \mathbf{p}, t) \prod_{\sigma=1}^3 \delta \left(x_{\sigma} - q_{\sigma} \cos \theta_{\sigma} - p_{\sigma} \frac{\sin \theta_{\sigma}}{m\omega} \right) d^3 q d^3 p, \quad (28)$$

$$M_{\text{cl}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t) = \int W_{\text{cl}}(\mathbf{q}, \mathbf{p}, t) \prod_{\sigma=1}^3 \delta (x_{\sigma} - \mu_{\sigma} q_{\sigma} - \nu_{\sigma} p_{\sigma}) d^3 q d^3 p, \quad (29)$$

are also independent on the gauge transformation. We use the designation \mathbf{x} instead of \mathbf{X} for distribution variable to point out that the Radon transformations (28) and (29) are made in the phase space with kinetic momentum \mathbf{p} .

Using the known correspondence rules [23, 26] between the operators acting on the Wigner function [6] (or on the distribution function) and the operators acting on the optical or symplectic tomograms

$$\begin{aligned} q_{\sigma} W(\mathbf{q}, \mathbf{p}) &\longleftrightarrow -\partial_{x_{\sigma}}^{-1} \partial_{\mu_{\sigma}} M(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) \longleftrightarrow (\sin \theta_{\sigma} \partial_{x_{\sigma}}^{-1} \partial_{\theta_{\sigma}} + x_{\sigma} \cos \theta_{\sigma}) w(\mathbf{x}, \boldsymbol{\theta}), \\ p_{\sigma} W(\mathbf{q}, \mathbf{p}) &\longleftrightarrow -\partial_{x_{\sigma}}^{-1} \partial_{\nu_{\sigma}} M(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) \longleftrightarrow m\omega (-\cos \theta_{\sigma} \partial_{x_{\sigma}}^{-1} \partial_{\theta_{\sigma}} + x_{\sigma} \sin \theta_{\sigma}) w(\mathbf{x}, \boldsymbol{\theta}), \\ \partial_{q_{\sigma}} W(\mathbf{q}, \mathbf{p}) &\longleftrightarrow \mu_{\sigma} \partial_{x_{\sigma}} M(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) \longleftrightarrow \cos \theta_{\sigma} \partial_{x_{\sigma}} w(\mathbf{x}, \boldsymbol{\theta}), \\ \partial_{p_{\sigma}} W(\mathbf{q}, \mathbf{p}) &\longleftrightarrow \nu_{\sigma} \partial_{x_{\sigma}} M(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) \longleftrightarrow \frac{\sin \theta_{\sigma}}{m\omega} \partial_{x_{\sigma}} w(\mathbf{x}, \boldsymbol{\theta}), \end{aligned} \quad (30)$$

where we introduced the designation [26] for inverse derivatives

$$\partial_{x_\sigma}^{-n} f(x_\sigma) = \frac{1}{(n-1)!} \int (x_\sigma - x'_\sigma)^{n-1} \Theta(x_\sigma - x'_\sigma) f(x'_\sigma) dx'_\sigma, \quad (31)$$

where $\Theta(x_\sigma - x'_\sigma)$ is a Heaviside step function, we can write Liouville equation (27) in the optical and the symplectic tomography representation

$$\begin{aligned} \partial_t w_{\text{cl}}(\mathbf{x}, \boldsymbol{\theta}, t) = & \left[\omega \sum_{j=1}^3 \left(\cos^2 \theta_j \partial_{\theta_j} - \frac{\sin 2\theta_j}{2} \{1 + x_j \partial_{x_j}\} \right) \right. \\ & + \frac{e}{mc} \sum_{\alpha, \beta, \gamma=1}^3 \varepsilon_{\alpha\beta\gamma} B_\gamma \left(\sin \theta_\sigma \partial_{x_\sigma}^{-1} \partial_{\theta_\sigma} + x_\sigma \cos \theta_\sigma, t \right) \left(\cos \theta_\beta \partial_{x_\beta}^{-1} \partial_{\theta_\beta} - x_\beta \sin \theta_\beta \right) \sin \theta_\alpha \partial_{x_\alpha} \\ & \left. - \frac{e}{m\omega} \sum_{j=1}^3 E_j \left(\sin \theta_\sigma \partial_{x_\sigma}^{-1} \partial_{\theta_\sigma} + x_\sigma \cos \theta_\sigma, t \right) \sin \theta_j \partial_{x_j} \right] w_{\text{cl}}(\mathbf{x}, \boldsymbol{\theta}, t), \end{aligned} \quad (32)$$

$$\begin{aligned} \partial_t M_{\text{cl}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t) = & \left[\frac{\boldsymbol{\mu}}{m} \partial_{\boldsymbol{\nu}} + \frac{e}{mc} \sum_{\alpha, \beta, \gamma=1}^3 \varepsilon_{\alpha\beta\gamma} B_\gamma \left(-\partial_{x_\sigma}^{-1} \partial_{\mu_\sigma}, t \right) \left(\partial_{x_\beta}^{-1} \partial_{\nu_\beta} \right) \nu_\alpha \partial_{x_\alpha} \right. \\ & \left. - e \sum_{j=1}^3 E_j \left(-\partial_{x_\sigma}^{-1} \partial_{\mu_\sigma}, t \right) \nu_j \partial_{x_j} \right] M_{\text{cl}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t), \end{aligned} \quad (33)$$

where $\varepsilon_{\alpha\beta\gamma}$ is the completely antisymmetric pseudo-tensor of 3rd rank (the Levi-Civita symbol).

Thus, we have gauge-independent equations (32,33) for gauge-independent classical tomograms $w_{\text{cl}}(\mathbf{x}, \boldsymbol{\theta}, t)$ and $M_{\text{cl}}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$.

As it is known, if we have the ensemble of non-interacting particles in the potential field, the generalized momentum of the particle is equal to its kinetic momentum, and the quantum analogue of the Liouville equation in this case is Moyal equation [29] for the Wigner function [6]

$$W(\mathbf{q}, \mathbf{P}, t) = \frac{1}{(2\pi\hbar)^3} \int \rho \left(\mathbf{q} - \frac{\mathbf{u}}{2}, \mathbf{q} + \frac{\mathbf{u}}{2}, t \right) \exp \left(\frac{i}{\hbar} \mathbf{u} \mathbf{P} \right) d^3 u, \quad (34)$$

which is converted into the Liouville equation when $\hbar \rightarrow 0$.

In the electro-magnetic field when $\mathbf{A} \neq 0$, the Moyal equation for the function (34) is written as follows:

$$\begin{aligned} \frac{\partial}{\partial t} W(\mathbf{q}, \mathbf{P}, t) = & \left[-\frac{\mathbf{P}}{m} \frac{\partial}{\partial \mathbf{q}} + \frac{2e}{\hbar} \text{Im} \varphi \left(\mathbf{q} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{P}}, t \right) + \frac{e^2}{mc^2 \hbar} \text{Im} \mathbf{A}^2 \left(\mathbf{q} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{P}}, t \right) \right. \\ & - \frac{2e}{mc\hbar} \text{Im} \left\{ \mathbf{A} \left(\mathbf{q} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{P}}, t \right) \left(\mathbf{P} - \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{q}} \right) \right\} \\ & \left. + \frac{e}{mc} \text{Re} \nabla_{\mathbf{q}} \mathbf{A} \left(\mathbf{q} \rightarrow \mathbf{q} + \frac{i\hbar}{2} \frac{\partial}{\partial \mathbf{P}}, t \right) \right] W(\mathbf{q}, \mathbf{P}, t). \end{aligned} \quad (35)$$

The function $W(\mathbf{q}, \mathbf{P}, t)$ is gauge-dependent, but if we take the classical limit $\hbar \rightarrow 0$ and change variables $\mathbf{p} = \mathbf{P} - \frac{e}{c} \mathbf{A}$ in (35), then this equation will be converted into gauge-independent Liouville equation (27). However, there is no contradiction here, because in the gauge transformation of the function $W(\mathbf{q}, \mathbf{p} + \frac{e}{c} \mathbf{A})$

$$\begin{aligned} W_{\text{c}} \left(\mathbf{q}, \mathbf{p} + \frac{e}{c} \mathbf{A} \right) = & \int W \left(\mathbf{q}, \mathbf{p}' + \frac{e}{c} \mathbf{A} \right) \exp \left\{ \frac{i}{\hbar} \mathbf{u} (\mathbf{p} - \mathbf{p}') \right\} \\ & \times \exp \left\{ \frac{ie}{c\hbar} \left[\chi \left(\mathbf{q} - \frac{\mathbf{u}}{2} \right) - \chi \left(\mathbf{q} + \frac{\mathbf{u}}{2} \right) + \mathbf{u} \nabla \chi(\mathbf{q}) \right] \right\} \frac{d^3 u d^3 p'}{(2\pi\hbar)^3} \end{aligned}$$

we can spread out the function $\chi(\mathbf{q} \pm \mathbf{u}/2)$ up to the first order $\chi(\mathbf{q} \pm \mathbf{u}/2) \approx \chi(\mathbf{q}) \pm \frac{1}{2}\mathbf{u}\nabla\chi(\mathbf{q})$ using a method of a stationary phase at $\hbar \rightarrow 0$. After that in the limit case we obtain

$$W_c\left(\mathbf{q}, \mathbf{p} + \frac{e}{c}\mathbf{A}\right) = \int W\left(\mathbf{q}, \mathbf{p}' + \frac{e}{c}\mathbf{A}\right) \delta(\mathbf{p} - \mathbf{p}') d^3p' = W\left(\mathbf{q}, \mathbf{p} + \frac{e}{c}\mathbf{A}\right),$$

that is the function $W\left(\mathbf{q}, \mathbf{p} + \frac{e}{c}\mathbf{A}\right)$ becomes gauge-independent.

Let us transform Moyal equation (35) to the optical and symplectic tomographic representations, in which the tomograms $w(\mathbf{X}, \boldsymbol{\theta}, t)$ and $M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ are defined from the Wigner function $W(\mathbf{q}, \mathbf{P}, t)$ with the same formulas (28) and (29), where the kinetic momentum \mathbf{p} should be replaced by the generalized momentum \mathbf{P} , and the variable \mathbf{x} should be replaced by \mathbf{X} to point out that the Radon transformations are being done in the phase space with generalized momentum. For this purpose we should use the same correspondence rules as (30). After calculations we can write the evolution equation for gauge-dependent optical tomogram as follows:

$$\begin{aligned} \partial_t w(\mathbf{X}, \boldsymbol{\theta}, t) &= \left[\omega \sum_{j=1}^3 \left(\cos^2 \theta_j \partial_{\theta_j} - \frac{1}{2} \sin 2\theta_j \left\{ 1 + X_j \partial_{X_j} \right\} \right) + \frac{2e}{\hbar} \text{Im} [\hat{\varphi}]_w \right. \\ &\quad \left. + \frac{e^2}{mc^2 \hbar} \text{Im} [\hat{\mathbf{A}}]_w^2 - \frac{2e}{mc \hbar} \text{Im} \left([\hat{\mathbf{A}}]_w [\hat{\mathbf{P}}]_w \right) + \frac{e}{mc} \text{Re} [\nabla_{\mathbf{q}} \mathbf{A}]_w \right] w(\mathbf{X}, \boldsymbol{\theta}, t), \end{aligned} \quad (36)$$

where

$$\begin{aligned} [\hat{A}_j]_w &= A_j([\hat{\mathbf{q}}]_w, t), \quad [\hat{\varphi}]_w = \varphi([\hat{\mathbf{q}}]_w, t), \\ [\nabla_{\mathbf{q}} \hat{\mathbf{A}}]_w &= \nabla_{\mathbf{q}} \mathbf{A}(\mathbf{q} \rightarrow [\hat{\mathbf{q}}]_w, t), \end{aligned}$$

and $[\hat{\mathbf{q}}]_w, [\hat{\mathbf{P}}]_w$ are position and generalized momentum operators in the optical tomographic representation [26],

$$[\hat{q}_\sigma]_w = \sin \theta_\sigma \partial_{\theta_\sigma} \partial_{X_\sigma}^{-1} + X_\sigma \cos \theta_\sigma + i \frac{\hbar \sin \theta_\sigma}{2m\omega} \partial_{X_\sigma}, \quad (37)$$

$$[\hat{P}_\sigma]_w = m\omega (-\cos \theta_\sigma \partial_{X_\sigma}^{-1} \partial_{\theta_\sigma} + X_\sigma \sin \theta_\sigma) - \frac{i\hbar}{2} \cos \theta_\sigma \partial_{X_\sigma}. \quad (38)$$

For the gauge-dependent symplectic tomogram we can write

$$\begin{aligned} \partial_t M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}, t) &= \left[\frac{\boldsymbol{\mu}}{m} \partial_{\boldsymbol{\nu}} + \frac{2e}{\hbar} \text{Im} [\hat{\varphi}]_M + \frac{e^2}{mc^2 \hbar} \text{Im} [\hat{\mathbf{A}}]_M^2 \right. \\ &\quad \left. - \frac{2e}{mc \hbar} \text{Im} \left([\hat{\mathbf{A}}]_M [\hat{\mathbf{P}}]_M \right) + \frac{e}{mc} \text{Re} [\nabla_{\mathbf{q}} \mathbf{A}]_M \right] M(\mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\nu}, t), \end{aligned} \quad (39)$$

where

$$\begin{aligned} [\hat{A}_j]_M &= A_j([\hat{\mathbf{q}}]_M, t), \quad [\hat{\varphi}]_M = \varphi([\hat{\mathbf{q}}]_M, t), \\ [\nabla_{\mathbf{q}} \hat{\mathbf{A}}]_M &= \nabla_{\mathbf{q}} \mathbf{A}(\mathbf{q} \rightarrow [\hat{\mathbf{q}}]_M, t), \end{aligned}$$

and $[\hat{\mathbf{q}}]_M, [\hat{\mathbf{P}}]_M$ are position and generalized momentum operators in the symplectic representation (see [23]),

$$[\hat{P}_\sigma]_M = -\partial_{X_\sigma}^{-1} \partial_{\nu_\sigma} - i(\hbar/2) \mu_\sigma \partial_{X_\sigma}, \quad [\hat{q}_\sigma]_M = -\partial_{X_\sigma}^{-1} \partial_{\mu_\sigma} + i(\hbar/2) \nu_\sigma \partial_{X_\sigma}. \quad (40)$$

Equations (36), (39) are gauge-invariant only under the condition of transformation of tomograms with general formula (14) with the kernel $G(z, \eta, z', \eta')$ defined by formula (18) or (21). In the classical limit $\hbar \rightarrow 0$ these equations, in general case, are not converted into equations (32), (33). The thing is that (32) and (36) are equations for distribution functions of different observables: $x_\sigma(\theta_\sigma) = q_\sigma \cos \theta_\sigma + p_\sigma \sin \theta_\sigma$ in the classical case (32); but $\hat{X}_\sigma(\theta_\sigma) = \hat{q}_\sigma \cos \theta_\sigma + \hat{P}_\sigma \sin \theta_\sigma$ in the quantum case (36). Analogously, (33) and (39) are equations for distribution functions of different observables $x_\sigma(\mu_\sigma, \nu_\sigma) = q_\sigma \mu_\sigma + p_\sigma \nu_\sigma$ and $\hat{X}_\sigma(\mu_\sigma, \nu_\sigma) = \hat{q}_\sigma \mu_\sigma + \hat{P}_\sigma \nu_\sigma$ respectively.

4 Gauge-independent tomographic quasiprobability representations

In the previous section we have shown that the evolution equations in the tomographic representations for the gauge-dependent tomograms in the classical limit $\hbar \rightarrow 0$ are not converted to the Liouville equation in the tomographic forms for gauge-independent tomograms of the classical distribution function.

Therefore, for the construction of quantum tomographic representations, in which the evolution equations would have been transformed to (32) and (33) when $\hbar \rightarrow 0$, we need to introduce gauge-independent quantum tomograms. This can be done with the help of a gauge-independent Wigner function obtained in [24],

$$W_g(\mathbf{q}, \mathbf{p}, t) = \frac{1}{(2\pi\hbar)^3} \int \exp \left(\frac{i}{\hbar} \mathbf{u} \left\{ \mathbf{p} + \frac{e}{c} \int_{-1/2}^{1/2} d\tau \mathbf{A}(\mathbf{q} + \tau \mathbf{u}, t) \right\} \right) \rho \left(\mathbf{q} - \frac{\mathbf{u}}{2}, \mathbf{q} + \frac{\mathbf{u}}{2}, t \right) d^3u, \quad (41)$$

where \mathbf{p} is a kinetic momentum.

Gauge-independent Moyal equation for this function has the form [30]:

$$\left\{ \partial_t + \frac{1}{m} (\mathbf{p} + \Delta \tilde{\mathbf{p}}) \partial_{\mathbf{q}} + e \left(\tilde{\mathbf{E}} + \frac{1}{mc} [(\mathbf{p} + \Delta \tilde{\mathbf{p}}) \times \tilde{\mathbf{B}}] \right) \partial_{\mathbf{p}} \right\} W_g(\mathbf{q}, \mathbf{p}, t) = 0, \quad (42)$$

where

$$\Delta \tilde{\mathbf{p}} = -\frac{e\hbar}{c} \frac{\partial}{\partial \mathbf{p}} \times \int_{-1/2}^{1/2} d\tau \tau \mathbf{B} \left(\mathbf{q} + i\hbar\tau \frac{\partial}{\partial \mathbf{p}}, t \right),$$

$$\tilde{\mathbf{E}} = \int_{-1/2}^{1/2} d\tau \mathbf{E} \left(\mathbf{q} + i\hbar\tau \frac{\partial}{\partial \mathbf{p}}, t \right), \quad \tilde{\mathbf{B}} = \int_{-1/2}^{1/2} d\tau \mathbf{B} \left(\mathbf{q} + i\hbar\tau \frac{\partial}{\partial \mathbf{p}}, t \right).$$

This equation in the classical limit $\hbar \rightarrow 0$ is converted into Liouville equation (27).

If we apply Radon transformations (28) and (29) to the Wigner function (41), we obtain gauge-independent optical $w_g(\mathbf{x}, \boldsymbol{\theta}, t)$ and symplectic $M_g(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ tomograms. Under such definitions the correspondence rules between operators acting on the tomograms and the Wigner function will be similar to the correspondence rules (30). Then, from equation (42) we find the evolution equation for the gauge-independent optical tomogram $w_g(\mathbf{x}, \boldsymbol{\theta}, t)$:

$$\begin{aligned} \partial_t w_g(\mathbf{x}, \boldsymbol{\theta}, t) = & \left[\omega \sum_{j=1}^3 \left(\cos^2 \theta_j \partial_{\theta_j} - \frac{1}{2} \sin 2\theta_j \{1 + x_j \partial_{x_j}\} \right) \right. \\ & - \frac{1}{m} \sum_{\alpha=1}^3 [\Delta \tilde{\mathbf{p}}_{\alpha}]_w \cos \theta_{\alpha} \partial_{x_{\alpha}} - \frac{e}{m\omega} \sum_{j=1}^3 [\tilde{\mathbf{E}}_j]_w \sin \theta_j \partial_{x_j} \Big) \\ & \left. + \frac{e}{mc} \sum_{\alpha, \beta, \gamma=1}^3 \varepsilon_{\alpha\beta\gamma} [\tilde{\mathbf{B}}_{\gamma}]_w \left(\cos \theta_{\beta} \partial_{x_{\beta}}^{-1} \partial_{\theta_{\beta}} - x_{\beta} \sin \theta_{\beta} - [\Delta \tilde{\mathbf{p}}_{\beta}]_w \right) \sin \theta_{\alpha} \partial_{x_{\alpha}} \right] w_g(\mathbf{x}, \boldsymbol{\theta}, t), \quad (43) \end{aligned}$$

where

$$[\Delta \tilde{\mathbf{p}}_{\alpha}]_w = -\frac{e}{mc\omega} \frac{\hbar}{i} \sum_{\beta, \gamma=1}^3 \varepsilon_{\alpha\beta\gamma} \sin \theta_{\beta} \partial_{x_{\beta}} \int_{-1/2}^{1/2} d\tau \tau \mathbf{B}_{\gamma} \left(\sin \theta_{\sigma} \partial_{x_{\sigma}}^{-1} \partial_{\theta_{\sigma}} + x_{\sigma} \cos \theta_{\sigma} + \frac{i\hbar\tau}{m\omega} \sin \theta_{\sigma} \partial_{x_{\sigma}}, t \right),$$

$$[\tilde{\mathbf{E}}]_w = \int_{-1/2}^{1/2} d\tau \mathbf{E} \left(\sin \theta_{\sigma} \partial_{x_{\sigma}}^{-1} \partial_{\theta_{\sigma}} + x_{\sigma} \cos \theta_{\sigma} + \frac{i\hbar\tau}{m\omega} \sin \theta_{\sigma} \partial_{x_{\sigma}}, t \right),$$

$$[\tilde{\mathbf{B}}]_w = \int_{-1/2}^{1/2} d\tau \mathbf{B} \left(\sin \theta_{\sigma} \partial_{x_{\sigma}}^{-1} \partial_{\theta_{\sigma}} + x_{\sigma} \cos \theta_{\sigma} + \frac{i\hbar\tau}{m\omega} \sin \theta_{\sigma} \partial_{x_{\sigma}}, t \right).$$

For symplectic tomogram $M_g(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ we obtain

$$\begin{aligned} \partial_t M_g(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t) &= \left[\frac{\boldsymbol{\mu}}{m} \partial_{\boldsymbol{\nu}} - \frac{1}{m} \sum_{\alpha=1}^3 [\Delta \tilde{\mathbf{p}}_{\alpha}]_M \mu_{\alpha} \partial_{x_{\alpha}} - e \sum_{j=1}^3 [\tilde{\mathbf{E}}_j]_M \nu_j \partial_{x_j} \right. \\ &\quad \left. + \frac{e}{mc} \sum_{\alpha, \beta, \gamma=1}^3 \varepsilon_{\alpha\beta\gamma} [\tilde{\mathbf{B}}_{\gamma}]_M \left(\partial_{x_{\beta}}^{-1} \partial_{\nu_{\beta}} - [\Delta \tilde{\mathbf{p}}_{\beta}]_M \right) \nu_{\alpha} \partial_{x_{\alpha}} \right] M_g(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t), \end{aligned} \quad (44)$$

where

$$\begin{aligned} [\Delta \tilde{\mathbf{p}}_{\alpha}]_M &= -\frac{e}{c} \frac{\hbar}{i} \sum_{\beta, \gamma=1}^3 \varepsilon_{\alpha\beta\gamma} \nu_{\beta} \partial_{x_{\beta}} \int_{-1/2}^{1/2} d\tau \tau \mathbf{B}_{\gamma} \left(-\partial_{x_{\sigma}}^{-1} \partial_{\mu_{\sigma}} + i\hbar \tau \nu_{\sigma} \partial_{x_{\sigma}}, t \right), \\ [\tilde{\mathbf{E}}]_M &= \int_{-1/2}^{1/2} d\tau \mathbf{E} \left(-\partial_{x_{\sigma}}^{-1} \partial_{\mu_{\sigma}} + i\hbar \tau \nu_{\sigma} \partial_{x_{\sigma}}, t \right), \\ [\tilde{\mathbf{B}}]_M &= \int_{-1/2}^{1/2} d\tau \mathbf{B} \left(-\partial_{x_{\sigma}}^{-1} \partial_{\mu_{\sigma}} + i\hbar \tau \nu_{\sigma} \partial_{x_{\sigma}}, t \right). \end{aligned}$$

As it should be, equations (43) and (44) in the classical limit $\hbar \rightarrow 0$ are converted into the equations (32) and (33).

Combining formulas (29) and (41) we can write

$$M_g(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \int \langle \mathbf{q} | \hat{\rho} | \mathbf{q}' \rangle \langle \mathbf{q}' | \hat{U}_{M_g}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) | \mathbf{q} \rangle d^3 q d^3 q',$$

where we introduced the designation for the matrix element of the corresponding dequantizer

$$\begin{aligned} \langle \mathbf{q}' | \hat{U}_{M_g}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) | \mathbf{q} \rangle &= \frac{1}{(2\pi\hbar)^3} \prod_{\sigma=1}^3 |\nu_{\sigma}|^{-1} \exp \left\{ \frac{i}{\hbar} (q'_{\sigma} - q_{\sigma}) \left[\frac{x_{\sigma}}{\nu_{\sigma}} - \frac{\mu_{\sigma}(q'_{\sigma} + q_{\sigma})}{2\nu_{\sigma}} \right. \right. \\ &\quad \left. \left. + \frac{e}{c} \int_{-1/2}^{1/2} d\tau A_{\sigma} \left(\frac{\mathbf{q}' + \mathbf{q}}{2} + \tau(\mathbf{q}' - \mathbf{q}) \right) \right] \right\}. \end{aligned} \quad (45)$$

From (45) we can see that $\hat{U}_{M_g}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is Hermitian and non-negative operator, consequently, the tomogram $M_g(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu})$ is real and non-negative.

From the structure of matrix element (45) and the fact, that the components of the kinetic momentum operator $\hat{\mathbf{p}} = \hat{\mathbf{P}} - \frac{e}{c} \mathbf{A}(\hat{\mathbf{q}})$ do not commute, it is possible to guess that the explicit expression for the dequantizer \hat{U}_{M_g} looks like

$$\hat{U}_{M_g}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \int \frac{d^3 k}{(2\pi)^3} \exp \left\{ i \sum_{\sigma=1}^3 k_{\sigma} \left[x_{\sigma} - \mu_{\sigma} \hat{q}_{\sigma} - \nu_{\sigma} \hat{P}_{\sigma} + \nu_{\sigma} \frac{e}{c} A_{\sigma}(\hat{\mathbf{q}}, t) \right] \right\} \quad (46)$$

Indeed, calculation of the matrix element of operator (46) gives the result (45).

Formula (46) permits to determine the corresponding quantizer as follows:

$$\hat{D}_{M_g}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \left(\frac{m\omega}{2\pi} \right)^3 \exp \left\{ i \sqrt{\frac{m\omega}{\hbar}} \sum_{\sigma=1}^3 \left[x_{\sigma} - \mu_{\sigma} \hat{q}_{\sigma} - \nu_{\sigma} \hat{P}_{\sigma} + \nu_{\sigma} \frac{e}{c} A_{\sigma}(\hat{\mathbf{q}}, t) \right] \right\}. \quad (47)$$

We can see that the dequantizer and the quantizer are gauge-invariant in the sense of transformation of type (7).

After calculations for the matrix element of (47) we obtain

$$\begin{aligned}
\langle \mathbf{q} | \hat{D}_{M_g}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) | \mathbf{q}' \rangle &= \left(\frac{m\omega}{2\pi} \right)^3 \exp \left\{ \frac{ie}{c\hbar} (\mathbf{q} - \mathbf{q}') \int_{-1/2}^{1/2} d\tau \mathbf{A} \left(\frac{\mathbf{q}' + \mathbf{q}}{2} + \tau(\mathbf{q} - \mathbf{q}') \right) \right\} \\
&\times \delta \left(\mathbf{q} - \mathbf{q}' - \boldsymbol{\nu} \sqrt{m\omega\hbar} \right) \exp \left\{ i\boldsymbol{\mu} \left[\boldsymbol{\nu} \frac{m\omega}{2} - \mathbf{q} \sqrt{\frac{m\omega}{\hbar}} \right] \right\} \\
&\times \prod_{\sigma=1}^3 \exp \left\{ ix_{\sigma} \sqrt{\frac{m\omega}{\hbar}} \right\}.
\end{aligned} \tag{48}$$

Using formulas (45) and (48) it is possible to check up that

$$\int \langle \mathbf{q}_2 | \hat{U}_{M_g}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) | \mathbf{q}_1 \rangle \langle \mathbf{q}'_1 | \hat{D}_{M_g}(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}) | \mathbf{q}'_2 \rangle d^3x d^3\mu d^3\nu = \delta(\mathbf{q}_1 - \mathbf{q}'_1) \delta(\mathbf{q}_2 - \mathbf{q}'_2).$$

It is obvious that the corresponding dequantizer and the quantizer for optical tomogram $w_g(\mathbf{x}, \boldsymbol{\theta}, t)$ are related with (46) and (47) as follows:

$$\begin{aligned}
\hat{U}_{w_g}(\mathbf{x}, \boldsymbol{\theta}) &= \hat{U}_{M_g} \left(x_{\sigma}, \mu_{\sigma} = \cos \theta_{\sigma}, \nu_{\sigma} = \frac{\sin \theta_{\sigma}}{m\omega} \right), \\
\hat{D}_{w_g}(\mathbf{x}, \boldsymbol{\theta}) &= \int \frac{|k_1| |k_2| |k_3|}{(m\omega)^3} \hat{D}_{M_g} \left(k_{\sigma} \sqrt{\frac{\hbar}{m\omega}} x_{\sigma}, \mu_{\sigma} = k_{\sigma} \sqrt{\frac{\hbar}{m\omega}} \cos \theta_{\sigma}, \nu_{\sigma} = k_{\sigma} \sqrt{\frac{\hbar}{m\omega}} \frac{\sin \theta_{\sigma}}{m\omega} \right) d^3k.
\end{aligned}$$

Due to the fact that the components of the operator $\hat{\mathbf{x}}(\boldsymbol{\theta})$ as well as $\hat{\mathbf{x}}(\boldsymbol{\mu}, \boldsymbol{\nu})$ do not commute, constructed in this section tomographic representations are not probability representations, but they are non-negative, normalized, and gauge-independent quasi-probability tomographic representations.

5 Gauge-independent probability representation

Unfortunately, introduced in the previous section gauge-independent tomographic functions $M_g(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ and $w_g(\mathbf{x}, \boldsymbol{\theta}, t)$ are not distribution functions of any physical observable. To make up for this shortcoming, we introduce the tomographic function $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ as the following map of the gauge-independent Wigner function:

$$\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t) = \int W_g(\mathbf{q}, \mathbf{p}, t) \delta(x - \boldsymbol{\mu}\mathbf{q} - \boldsymbol{\nu}\mathbf{p}) d^3q d^3p. \tag{49}$$

It is evident that $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ is a distribution function of the physical observable $\hat{x}(\boldsymbol{\mu}, \boldsymbol{\nu}) = \boldsymbol{\mu}\hat{\mathbf{q}} + \boldsymbol{\nu}\hat{\mathbf{p}}$, which is a scalar product of two 6-dimensional vectors $(\boldsymbol{\mu}, \boldsymbol{\nu})$ and $(\hat{\mathbf{q}}, \hat{\mathbf{p}})$. The quantity $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)dx$ is the probability to have the value of the scalar operator $\hat{x}(\boldsymbol{\mu}, \boldsymbol{\nu})$ within the interval between x and $x + dx$ at fixed time t and fixed vector $(\boldsymbol{\mu}, \boldsymbol{\nu})$.

The map inverse to (49) has, obviously, the form:

$$W_g(\mathbf{q}, \mathbf{p}, t) = \left(\frac{m\omega}{4\pi^2\hbar} \right)^3 \int \mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t) \exp \left\{ i\sqrt{\frac{m\omega}{\hbar}} (x - \boldsymbol{\mu}\mathbf{q} - \boldsymbol{\nu}\mathbf{p}) \right\} dx d^3\mu d^3\nu. \tag{50}$$

Combining formulas (49) and (41) we obtain the expression for the matrix element of the dequantizer operator $\hat{U}_{\mathfrak{M}}(x, \boldsymbol{\mu}, \boldsymbol{\nu})$ for this representation

$$\begin{aligned}
\langle \mathbf{q}' | \hat{U}_{\mathfrak{M}}(x, \boldsymbol{\mu}, \boldsymbol{\nu}) | \mathbf{q} \rangle &= \frac{1}{2\pi\hbar|\nu_3|} \delta \left(\frac{\nu_1}{\nu_3} (q'_3 - q_3) - (q'_1 - q_1) \right) \delta \left(\frac{\nu_2}{\nu_3} (q'_3 - q_3) - (q'_2 - q_2) \right) \\
&\times \exp \left\{ \frac{i}{\hbar} \left[x \frac{q'_3 - q_3}{\nu_3} - \mu_1 \frac{q_1'^2 - q_1^2}{2\nu_1} - \mu_2 \frac{q_2'^2 - q_2^2}{2\nu_2} - \mu_3 \frac{q_3'^2 - q_3^2}{2\nu_3} \right] \right\} \\
&\times \exp \left\{ \frac{ie}{c\hbar} (\mathbf{q}' - \mathbf{q}) \int_{-1/2}^{1/2} d\tau \mathbf{A} \left(\frac{\mathbf{q}' + \mathbf{q}}{2} + \tau(\mathbf{q}' - \mathbf{q}) \right) \right\}.
\end{aligned} \tag{51}$$

Taking into account expressions of dequantizer operators in previous sections and matrix element (51) we can write the explicit expression for the gauge-invariant dequantizer $\hat{U}_{\mathfrak{M}}(x, \boldsymbol{\mu}, \boldsymbol{\nu})$

$$\hat{U}_{\mathfrak{M}}(x, \boldsymbol{\mu}, \boldsymbol{\nu}) = \int \frac{dk}{2\pi} \exp \left\{ ik \left[x - \boldsymbol{\mu} \hat{\mathbf{q}} - \boldsymbol{\nu} \hat{\mathbf{p}} + \frac{e}{c} \boldsymbol{\nu} \mathbf{A}(\hat{\mathbf{q}}) \right] \right\}. \quad (52)$$

Then the quantizer $D_{\mathfrak{M}}(x, \boldsymbol{\mu}, \boldsymbol{\nu})$, obviously, equals

$$\hat{D}_{\mathfrak{M}}(x, \boldsymbol{\mu}, \boldsymbol{\nu}) = \left(\frac{m\omega}{2\pi} \right)^3 \exp \left\{ i \sqrt{\frac{m\omega}{\hbar}} \left[x - \boldsymbol{\mu} \hat{\mathbf{q}} - \boldsymbol{\nu} \hat{\mathbf{p}} + \frac{e}{c} \boldsymbol{\nu} \mathbf{A}(\hat{\mathbf{q}}) \right] \right\}, \quad (53)$$

and the calculation of it's matrix element gives rise to the following:

$$\begin{aligned} \langle \mathbf{q} | \hat{D}_{\mathfrak{M}}(x, \boldsymbol{\mu}, \boldsymbol{\nu}) | \mathbf{q}' \rangle &= \left(\frac{m\omega}{2\pi} \right)^3 \exp \left\{ \frac{ie}{c\hbar} (\mathbf{q} - \mathbf{q}') \int_{-1/2}^{1/2} d\tau \mathbf{A} \left(\frac{\mathbf{q}' + \mathbf{q}}{2} + \tau (\mathbf{q} - \mathbf{q}') \right) \right\} \\ &\times \delta \left(\mathbf{q} - \mathbf{q}' - \boldsymbol{\nu} \sqrt{m\omega\hbar} \right) \exp \left\{ i \boldsymbol{\mu} \left[\boldsymbol{\nu} \frac{m\omega}{2} - \mathbf{q} \sqrt{\frac{m\omega}{\hbar}} \right] + ix \sqrt{\frac{m\omega}{\hbar}} \right\}. \end{aligned} \quad (54)$$

The correspondence rules (30) for representation $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu})$ acquire a slightly modernized form

$$\begin{aligned} q_{\sigma} W_{\mathbf{g}}(\mathbf{q}, \mathbf{p}) &\longleftrightarrow -\partial_x^{-1} \partial_{\mu_{\sigma}} \mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}), \\ p_{\sigma} W_{\mathbf{g}}(\mathbf{q}, \mathbf{p}) &\longleftrightarrow -\partial_x^{-1} \partial_{\nu_{\sigma}} \mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}), \\ \partial_{q_{\sigma}} W_{\mathbf{g}}(\mathbf{q}, \mathbf{p}) &\longleftrightarrow \mu_{\sigma} \partial_x \mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}), \\ \partial_{p_{\sigma}} W_{\mathbf{g}}(\mathbf{q}, \mathbf{p}) &\longleftrightarrow \nu_{\sigma} \partial_x \mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}). \end{aligned} \quad (55)$$

With the help of (55) equation (42) is transformed to the evolution equation for the tomogram \mathfrak{M}

$$\begin{aligned} \partial_t \mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t) &= \left[\frac{\boldsymbol{\mu}}{m} \partial_{\boldsymbol{\nu}} - \frac{1}{m} \sum_{\alpha=1}^3 [\Delta \tilde{\mathbf{p}}_{\alpha}]_{\mathfrak{M}} \mu_{\alpha} \partial_x - e \sum_{j=1}^3 [\tilde{\mathbf{E}}_j]_{\mathfrak{M}} \nu_j \partial_x \right. \\ &\left. + \frac{e}{mc} \sum_{\alpha, \beta, \gamma=1}^3 \varepsilon_{\alpha\beta\gamma} [\tilde{\mathbf{B}}_{\gamma}]_{\mathfrak{M}} \left(\partial_{\nu_{\beta}} - [\Delta \tilde{\mathbf{p}}_{\beta}]_{\mathfrak{M}} \partial_x \right) \nu_{\alpha} \right] \mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t), \end{aligned} \quad (56)$$

where

$$\begin{aligned} [\Delta \tilde{\mathbf{p}}_{\alpha}]_M &= -\frac{e\hbar}{c} \sum_{\beta, \gamma=1}^3 \varepsilon_{\alpha\beta\gamma} \nu_{\beta} \partial_x \int_{-1/2}^{1/2} d\tau \tau \mathbf{B}_{\gamma} \left(-\partial_x^{-1} \partial_{\mu_{\sigma}} + i\hbar \tau \nu_{\sigma} \partial_x, t \right), \\ [\tilde{\mathbf{E}}]_M &= \int_{-1/2}^{1/2} d\tau \mathbf{E} \left(-\partial_x^{-1} \partial_{\mu_{\sigma}} + i\hbar \tau \nu_{\sigma} \partial_x, t \right), \\ [\tilde{\mathbf{B}}]_M &= \int_{-1/2}^{1/2} d\tau \mathbf{B} \left(-\partial_x^{-1} \partial_{\mu_{\sigma}} + i\hbar \tau \nu_{\sigma} \partial_x, t \right). \end{aligned}$$

In the limit case $\hbar \rightarrow 0$ we get the classical equation

$$\begin{aligned} \partial_t \mathfrak{M}_{\text{cl}}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t) &= \left[\frac{\boldsymbol{\mu}}{m} \partial_{\boldsymbol{\nu}} + \frac{e}{mc} \sum_{\alpha, \beta, \gamma=1}^3 \varepsilon_{\alpha\beta\gamma} B_{\gamma} \left(-\partial_x^{-1} \partial_{\mu_{\sigma}}, t \right) \nu_{\alpha} \partial_{\nu_{\beta}} \right. \\ &\left. - e \sum_{j=1}^3 E_j \left(-\partial_x^{-1} \partial_{\mu_{\sigma}}, t \right) \nu_j \partial_x \right] \mathfrak{M}_{\text{cl}}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t), \end{aligned} \quad (57)$$

which is the Liouville equation in corresponding representation. Thus, we have constructed the gauge-independent probability representation having the clear physical meaning and the classical limit.

The density matrix $\rho(\mathbf{q}, \mathbf{q}', t)$ depends on time and on six spatial variables, while the tomogram $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ depends on time, on one spatial variable, and on six tomography parameters. But the number of these parameters can be reduced by one if we take into account that the tomogram $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ is a homogeneous function in the sense

$$\mathfrak{M}(rx, r\boldsymbol{\mu}, r\boldsymbol{\nu}, t) = |r|^{-1} \mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t).$$

Therefore, in the 6-dimensional space $(\boldsymbol{\mu}, \tilde{\boldsymbol{\nu}}) = (\boldsymbol{\mu}, m\omega\boldsymbol{\nu})$ one can, for instance, pass to the unit sphere and reduce the number of variables introducing the new tomogram $\mathfrak{w}(x, \boldsymbol{\xi}, t)$ as follows:

$$\mathfrak{w}(x, \boldsymbol{\xi}, t) = \mathfrak{M}\left(x, \boldsymbol{\mu}(\boldsymbol{\xi}), \frac{\tilde{\boldsymbol{\nu}}(\boldsymbol{\xi})}{m\omega}, t\right) = \int W_g(\mathbf{q}, \mathbf{p}, t) \delta\left(x - \boldsymbol{\mu}(\boldsymbol{\xi})\mathbf{q} - \tilde{\boldsymbol{\nu}}(\boldsymbol{\xi})\frac{\mathbf{p}}{m\omega}\right) d^3q d^3p, \quad (58)$$

where $\boldsymbol{\xi}$ is a 5-dimensional vector of directional angles in the 6-dimensional space and

$$\begin{pmatrix} \boldsymbol{\mu}(\boldsymbol{\xi}) \\ \tilde{\boldsymbol{\nu}}(\boldsymbol{\xi}) \end{pmatrix} = \begin{pmatrix} \mu_1(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \\ \mu_2(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \\ \mu_3(\xi_2, \xi_3, \xi_4, \xi_5) \\ \nu_1(\xi_3, \xi_4, \xi_5) \\ \nu_2(\xi_4, \xi_5) \\ \nu_3(\xi_5) \end{pmatrix} = \begin{pmatrix} \sin \xi_1 \sin \xi_2 \sin \xi_3 \sin \xi_4 \sin \xi_5 \\ \cos \xi_1 \sin \xi_2 \sin \xi_3 \sin \xi_4 \sin \xi_5 \\ \cos \xi_2 \sin \xi_3 \sin \xi_4 \sin \xi_5 \\ \cos \xi_3 \sin \xi_4 \sin \xi_5 \\ \cos \xi_4 \sin \xi_5 \\ \cos \xi_5 \end{pmatrix}. \quad (59)$$

In physical meaning $\mathfrak{w}(x, \boldsymbol{\xi}, t)dx$ is the probability of the system to have the projection of the vector $(\mathbf{q}, \mathbf{p}/m\omega)$ on the direction of the unit vector (59) within the interval between x and $x + dx$.

The inverse transformation $\mathfrak{w}(x, \boldsymbol{\xi}, t) \rightarrow W_g(\mathbf{q}, \mathbf{p}, t)$ has, obviously, the form:

$$\begin{aligned} W_g(\mathbf{q}, \mathbf{p}, t) &= (4\pi^2 m\omega)^{-3} \int \mathfrak{w}(x, \boldsymbol{\xi}, t) \exp\left\{ir\left(x - \boldsymbol{\mu}(\boldsymbol{\xi})\mathbf{q} - \tilde{\boldsymbol{\nu}}(\boldsymbol{\xi})\frac{\mathbf{p}}{m\omega}\right)\right\} \\ &\times r^5 \sin \xi_2 \sin^2 \xi_3 \sin^3 \xi_4 \sin^4 \xi_5 dx dr d^5 \xi. \end{aligned} \quad (60)$$

So, the tomogram $\mathfrak{w}(x, \boldsymbol{\xi}, t)$ also contains all available information about the state of the system under study, but it depends on the same number of variables as the density matrix, and it is gauge-independent.

For the function $\mathfrak{w}(x, \boldsymbol{\xi}, t)$ it is also possible to write the evolution equation, and it is possible to reduce the number of variables of $\mathfrak{M}(x, \boldsymbol{\mu}, \boldsymbol{\nu}, t)$ by a more symmetrical method different from (58) - (59), but it may be the subject of future publications.

6 Conclusion

In conclusion we point out that the evolution equation of a tomogram of the state of quantum system, as well as the appropriate Moyal equation possess a gauge invariance. But the optical and symplectic tomograms in their determination with the help of gauge-independent dequantizers (15) and (19) do not possess the gauge independence and are converted by the integral transformation (14) with the kernel of type (13) dependent on the quantizer and dequantizer operators, and the gauge function χ .

Contrary to the quantum case, optical and symplectic tomograms of classical distribution function in the phase space with kinetic momentum possess of the gauge independence. Therefore, in the electromagnetic field the evolution equations (36) and (39) for gauge-independent tomograms do not have the classical limit (32) and (33) when $\hbar \rightarrow 0$. This quality differs from the quality of the Moyal equation, which is gauge-dependent but, nevertheless, has the gauge-independent Liouville equation as the classical limit.

To decide this problem we introduced the gauge-independent optical and symplectic tomographic quasi-distributions and tomographic probability distributions, and obtained gauge-independent evolution equations for them, which in the classical limit are converted to the Liouville equation in corresponding tomographic representations.

References

- [1] L. D. Landau and E. M. Lifshitz, *Field theory* (Pergamon, Oxford, 1997).
- [2] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics* (Pergamon, Oxford, 1982).
- [3] C. N. Yang and R. Mills, Phys. Rev. **96**, 191 (1954).
- [4] L. D. Landau and E. M. Lifshitz, *Quantum mechanics* (Pergamon, Oxford, 1997).
- [5] E. Schrodinger, Ann. Physik **79**, 361, 489 (1926).
- [6] E. Wigner, Phys. Rev. **40**, 749 (1932).
- [7] D. I. Blohintsev, J. Phys. USSA **2**, 71 (1940).
- [8] R. J. Glauber, Phys. Rev. Lett. **10**, 84 (1963).
- [9] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
- [10] K. Husimi, Proc. Phys.-Math. Soc. Japan **22**, 264 (1940).
- [11] S. Mancini, V. I. Man'ko, and P. Tombesi, Phys. Lett. A **213**, 1 (1996).
- [12] A. Ibort, V. I. Man'ko, G. Marmo, A. Simoni, and F. Ventriglia, Phys. Scr. **79**, 065013 (2009).
- [13] J. Bertrand and P. Bertrand, Found. Phys. **17**, 397 (1987).
- [14] K. Vogel and H. Risken, Phys. Rev. A **40**, 2847 (1989).
- [15] S. Mancini, V. I. Man'ko, and P. Tombesi, Quantum Semiclass. Opt. **7**, 615 (1995).
- [16] V. I. Man'ko and O. V. Man'ko, J. Exp. Theor. Phys. **85**, 430 (1997).
- [17] V. V. Dodonov and V. I. Man'ko, Phys. Lett. A **229**, 335 (1997).
- [18] O. V. Man'ko, V. I. Man'ko, and G. Marmo, J. Phys. A **35**, 699 (2002).
- [19] O. V. Man'ko, V. I. Man'ko, G. Marmo, and P. Vitale, Phys. Lett. A **360**, 522 (2007).
- [20] R. L. Stratonovich, Sov. Phys. JETP **4**, 891 (1957).
- [21] F. Lizzi and P. Vitale, SIGMA **10**, 086 (2014).
- [22] C. K. Zachos, D. B. Fairlie, and T. L. Curtrigh, *Quantum Mechanics in Phase Space: an Overview with Selected Papers* (Singapore: World Scientific, 2005).
- [23] Ya. A. Korennoy and V. I. Man'ko, J. Russ. Laser Res. **32**, 74 (2011).
- [24] R. L. Stratonovich, Sov. Phys. – Dokl. **1**, 414 (1956).
- [25] Ya. A. Korennoy and V. I. Man'ko, J. Russ. Laser Res. **32**, 338 (2011).

- [26] G. G. Amosov, Ya. A. Korennoy, and V. I. Man'ko, Phys. Rev. A **85**, 052119 (2012).
- [27] Ya. A. Korennoy and V. I. Man'ko, e-print arXiv:quant-ph/1412.7873 (2014).
- [28] Ya. A. Korennoy and V. I. Man'ko, e-print arXiv: quant-ph/1508.05978 (2015).
- [29] J. E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).
- [30] O. T. Serimaa, J. Javanainen, and S. Varró, Phys. Rev. A **33**, 2913 (1986).